Forced Monte Carlo Simulation Strategy for the Design of Maintenance Plans with Multiple Inspections

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Abstract: A maintenance problem can be regarded as an optimization task, where the solution is a trade-off between the costs associated with inspection and repair activities and the benefits related to the faultless operation of the infrastructure. The optimization aims at minimizing the total cost while tuning some parameters, such as the number, time, and quality of inspections. Due to the unavoidable uncertainties, the expected cost of maintenance and failure can only be estimated by assessing the reliability of the system. The problem is, therefore, formulated as a time-variant reliability-based optimization, where both objective and constraint functions require the assessment of reliability with time. This paper proposes an efficient general numerical technique to solve this problem by means of just one single reliability analysis, while explicitly taking the diverse forms of uncertainty into account. The technique is generally applicable to any problem where the ageing or damage propagation process is known by means of input-output relationships, which apply to a great number of the cases. This technique exploits a Monte Carlo strategy derived from the concept of forced simulation, which significantly increases the efficiency of computing the optimal solution. The efficiency and accuracy of the proposed approach is shown by means of an example involving a fatigue-prone weld in a bridge girder. **DOI: 10.1061/AJRUA6.0000868.** *© 2016 American Society of Civil Engineers.*

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Introduction

Preventive maintenance practice can be extremely cost effective for mitigating damage accumulation of civil infrastructures. In fact, inspection and repair activities may prevent loss of serviceability or even partial collapse. However, making decisions as to whether and when inspections should be performed is a very complex task because in real-scale engineering systems, scheduling maintenance activities often involves multiple conflicting optimization criteria (Zio et al. 2009). Moreover, the realistic quantification of costs associated with inspections, repair and failure (i.e., loss of serviceability), requires the explicit consideration of the unavoidable uncertainties arising from the damage-propagation process and from the inspection and repair activities. Reliability-based optimization methods and techniques, as described, e.g., in Jensen (2002), are invoked to solve the problem.

Due to the explicit consideration of uncertainties, the design of maintenance activities is an optimization task that requires the assessment of reliability, where number, times, and quality of inspections are the design variables and the total cost is the objective function. For the formulation of and solution to time-dependent reliability-based optimization problems, see, e.g., Patelli et al. (2011) and Valdebenito and Schuëller (2010a). The assessment of reliability both in the objective and in the constraint functions and the consideration of multiple inspections make this a stochastic discrete optimization problem, which is among the most complicated in the field of optimization (Wright and Nocedal 1999).

This paper proposes a general methodology for the efficient solution of the time-variant reliability-based maintenance optimization problem, which is applicable to any case in which the damage propagation law is known as the input-output relationship. The methodology is derived from the concept of forced Monte Carlo (MC) simulation used to evaluate the availability of plants (Zio and Marseguerra 2002), and it is exploited to efficiently assess the time-variant reliability conditional to the inspection outcomes, requiring only the execution of computationally inexpensive functions. There are no restrictions in terms of number of inspections and number of uncertain parameters using the proposed methodology.

Advantages of the Proposed Methodology

Two main advantages are identified, which make the proposed methodology particularly efficient:

- Only one full reliability analysis is required to estimate the failure probability until the mission time (or time of interest). In practice, the samples are generated and evaluated on the full model only once to estimate the failure probability at any inspection time. This turns out to be very useful because during the maintenance optimization process the reliability has to be assessed many times to find the optimal inspection time to perform inspections.
- The proposed methodology can be easily parallelized, allowing the efficient evaluation of the cost function.

The methodology can be easily extended to also include epistemic uncertainties without adding substantial computational load. This paper develops the basis for robust maintenance and sets the focus on the numerical strategy to solve the time-variant reliability-based optimization problem with multiple inspections.

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Optimization of Maintenance Costs

Given a system that evolves in time $\mathcal{S}(t)$, a mission time T_M, which is the time until the system is required to function as specified, and a number of inspections N performed at times $\boldsymbol{t}^{\text{insp}} \in \mathbb{R}^N$, the maintenance problem is formulated as an optimization task in which both objective and constraints require the evaluation of the reliability, $r(t)$. Three main different costs can be identified:

- Manufacturing (or initial) costs, C_0;
- Costs due to inspection and repair, $C_I + C_R$; and
- Costs of failure, C_F.

It is assumed that manufacturing costs are deterministic because they are linked to construction and usage of materials. The costs of repair and failure are expected values, $E[\cdot]$ because they are obtained from the estimation of repair and failure probability, respectively (Valdebenito and Schuëller 2010b).

Costs due to Inspections and Repair

The cost due to inspections depends on inspection quality, q, and the inspection times, $\boldsymbol{t}^{\text{insp}}$, and can be expressed as

$$E[C_I(q, \boldsymbol{t}^{\text{insp}})] = c_I q \eta(\boldsymbol{t}^{\text{insp}}) \tag{1}$$

where c_I = fixed unit cost; and q [Eq. (3)] quantifies the quality of inspections. In Eq. (1) the function

$$\eta(t) = \frac{1}{(1+s)^t} \tag{2}$$

is a discount function of annual interest rate, s. Eq. (2) takes into account that the interests are compounded annually. For example, the amount of money C invested today will be equivalent to $C(1+s)^t$ after t years if the interest is compounded annually, where $(1+s)^t$ is also called the accumulation factor. Therefore, the present value of an investment done t years in the future must be discounted by the amount $1/(1+s)^t$. Costs due to repair occur only if repair takes place, thus they depend on the probability of repair, $p_R(q,t)$, which is linked to the probability of detecting the damage within an inspection, POD, which in turns depends on the inspection quality, q, and on the level of damage, $D(t)$. For example, as a means of controlling damage associated with crack propagation, in fatigue-prone metallic components nondestructive inspection (NDI) techniques can be used. NDI techniques have an associated probability of detection (Zheng and Ellingwood 1998), which can be modeled as

$$\text{POD}(t) = (1 - p_0)(1 - e^{q[f_1 - f_2 D(t)]}) \tag{3}$$

where p_0 = probability of not detecting a large crack, while f_1 and f_2 are parameters that depend on the specific NDI technique. The probability of detection is calculated as the product of two factors: the probability of detecting a large crack, $1 - p_0$, times a coefficient $1 - e^{q[f_1 - f_2 D(t)]} \in [0,1]$, which is an increasing function of the state of damage, $D(t)$. The cost of repair can thus be expressed as

$$E[C_R(q, \boldsymbol{t}^{\text{insp}})] = c_R p_R(q, \boldsymbol{t}^{\text{insp}}) \eta(\boldsymbol{t}^{\text{insp}}) \tag{4}$$

where c_R = fixed unit cost. In some cases, the unit cost of repair can be very small or sometimes negligible compared with the cost of inspection. In fact, repair takes place contextually with inspection and it might not require a significant additional usage of resources.

Costs of Failure

The cost of failure depends on the quality, q, as well as on the state of damage, $D(t)$. Here, the expected failure cost can be expressed as

$$E[C_F(q, \boldsymbol{t}^{\text{insp}}, t)] = c_F p_F(q, \boldsymbol{t}^{\text{insp}}, t) \tag{5}$$

where c_F = fixed unit cost associated with failure, partial collapse, or unavailability; and $p_F(q, \boldsymbol{t}^{\text{insp}}, t)$ = failure probability, calculated as in the next section. The failure probability depends on both the inspection times, $\boldsymbol{t}^{\text{insp}}$, and on the time when the reliability is assessed.

Total Costs

The total cost, therefore, is

$$E[C_T] = C_0 + E[C_I(q, \boldsymbol{t}^{\text{insp}})] + E[C_R(q, \boldsymbol{t}^{\text{insp}})] + E[C_F(q, \boldsymbol{t}^{\text{insp}}, t)] \tag{6}$$

while the total cost of maintenance is

$$E[C_M] = E[C_I(q, \boldsymbol{t}^{\text{insp}})] + E[C_R(q, \boldsymbol{t}^{\text{insp}})] + E[C_F(q, \boldsymbol{t}^{\text{insp}}, t)] \tag{7}$$

Formulation of the Optimization Problem

The maintenance problem can be generally formulated as a constrained optimization problem in which the constraint represents the limit state safety level with which the system has to comply. Here, the following formulation of the optimization problem is considered:

$$\begin{aligned} &\underset{q \in \mathbb{R}^+, \boldsymbol{t}^{\text{insp}} \in [0, T_M]^N}{\text{minimize}} \; E[C_M(q, \boldsymbol{t}^{\text{insp}}, t)] \\ &\text{subject to} \;\; p_F(q, \boldsymbol{t}^{\text{insp}}, t) \le p_F^{\text{critic}} \end{aligned} \tag{8}$$

where p_F^{critic} is determined by a prescribed limit state safety level. The problem of Eq. (8) is addressed using the penalty function

$$\psi(c) = 1 - e^{\alpha|\min(0,c)|} \tag{9}$$

which is a function of the constraint

$$c = -\log[p_F(q, \boldsymbol{t}^{\text{insp}}, t)] + \log(p_F^{\text{critic}}) \tag{10}$$

where the constraint is satisfied if $c > 0$. The problem of Eq. (8) can thus be reformulated into an equivalent unconstrained problem as

$$\underset{q \in \mathbb{R}^+, \boldsymbol{t}^{\text{insp}} \in [0, T_M]^N}{\text{minimize}} \; E[C_M(q, \boldsymbol{t}^{\text{insp}}, t)] + g\psi(c) \tag{11}$$

where g = penalty factor, which value can be chosen knowing the order of magnitude of the minimum value of the objective function.

Time-Variant Reliability and Failure Probability Assessment

The damage can be expressed as a function, $D = D(\boldsymbol{\theta}, t) = D_{\boldsymbol{\theta}}(t)$, of some input parameters $\boldsymbol{\theta}$, that can be used to quantify the level of damage. For example, damage may manifest in the form of fatigue, where the model is represented by the Paris-Erdogan's law (Paris and Erdogan 1963), and $\boldsymbol{\theta}$ includes the initial crack length (initial condition), the stress range, the shape factors, the crack length ratio, and any other coefficients of the damage law. The time-variant reliability is obtained via definition of a critical threshold of damage, $D_{\boldsymbol{\theta}}^{\text{thres}}$, as

$$r(t) = 1 - P[D_{\boldsymbol{\theta}}(t) \ge D_{\boldsymbol{\theta}}^{\text{thres}}] \tag{12}$$

where the threshold, $D_{\boldsymbol{\theta}}^{\text{thres}}$, and the damage level, $D_{\boldsymbol{\theta}}(t)$, represent the capacity and the demand of the system, respectively. Both $D_{\boldsymbol{\theta}}^{\text{thres}}$

and $D_{\boldsymbol{\theta}}(t)$ are uncertain quantities with associated probability distribution functions. The time-variant reliability is obtained as

$$r(t) = 1 - \int_{D(\boldsymbol{\theta},t)\geq D^{\text{thres}}} h(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{13}$$

where h = joint density function of the random parameters $\boldsymbol{\theta}$. For simplicity, the capacity $D_{\boldsymbol{\theta}}^{\text{thres}}$ has been included in the vector of parameters, $\boldsymbol{\theta}$. By means of the Monte Carlo method the time-variant failure probability, $p_F(t)$, can be calculated as

$$p_F(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathcal{I}(\boldsymbol{\theta}, t) h(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{14}$$

where $\mathcal{I}(\boldsymbol{\theta}, t) \in \{0, 1\}$ = indicator function, which is 1 only if $D(\boldsymbol{\theta}, t) \geq D^{\text{thres}}$.

Formulation of the Maintenance Problem

The maintenance problem requires the evaluation of the reliability, $r(t)$, over the period of time, $t \in [0, T_M]$. With no inspections ($N = 0$), the problem can be solved by assessing the reliability as in Eq. (13). When inspections are considered, i.e., $N > 0$, the reliability of the system is conditional on the inspection outcomes. In fact, after an inspection the system or component can be regenerated as repair activities may take place. Therefore, the inspection outcomes and eventual repair need to be taken into account by computing the conditional reliability.

The optimal inspection time is naturally between the following two limiting cases. If inspections are performed too early, $\boldsymbol{t}^{\text{insp}} \ll T_M$, nearly no damage will be found, and hence no repair will take place. As a consequence the reliability will only be improved marginally, or even not improved at all. On the other side, if inspections are done too late, $\boldsymbol{t}^{\text{insp}} \simeq T_M$, the probability of detection would be large (because directly related to the level of damage), but it is likely that the system will have already failed, thus the inspection will not be effective. The reliability is a function of two different times: the actual time t, when the reliability is assessed, and the inspection time, $\boldsymbol{t}^{\text{insp}}$, when the inspections have been performed. In general, i.e., when N inspections are considered, the reliability is given by the conditional probability

$$r(t) = 1 - P[D(t) \geq D^{\text{thres}} | t_1^{\text{insp}} < t_2^{\text{insp}} < \dots t_N^{\text{insp}} < t] \tag{15}$$

Here, the focus is on assessing the reliability at a fixed time point, for example at the mission time $t = T_M$.

Assumptions

In order to illustrate the procedure, two assumptions are considered for simplicity, but without restricting the generality of the approach:

- It is assumed that any inspection is followed by only two outcomes: either the flaw is detected or it is not. If a flaw is detected repair takes place, which is assumed to be perfect, i.e., after repair $D(t^{\text{insp}}) = 0$. In other words, if a component is repaired, it is assumed that further chances of failure for that specific component are zero.
- Only preventive maintenance is considered. If the critical threshold is exceeded at the time of inspection, the component cannot be repaired. That is, if failure has occurred, repair actions will not take place.

Repair and failure events are closely related because they are both linked to the state of damage. For instance, if the damage is close to the critical threshold, it is very likely that either the failure or the repair event occurs. These assumptions can be easily relaxed.

Classification of Events and Total Failure Probability

In order to calculate the reliability as defined in Eq. (15), mutually exclusive events are classified and combined. Among all of the possible events four main classes are identified:

- The failure events, $F_i = [D(t_i^{\text{insp}}) > D^{\text{thres}}]$, at the time of the ith inspection;
- The failure event, $F_t = [D(t) > D^{\text{thres}}]$, at the evaluation time t;
- The repair or detection event, $R_i = [\delta(t_i^{\text{insp}}) = 1]$, at the time of the ith inspection; and
- The event $\overline{R_i} = [\delta(t_i^{\text{insp}}) = 0]$, i.e., the event of nonrepair or nondetection.

where δ is a binary random variable to characterize the outcome of inspections as will be explained in the next section.

The failure event, F, given that N inspections are performed, can be expressed by means of set operations (of union and intersection) among events.

The failure event is represented as a combination of mutually exclusive events

$$F = \bigcup_{j=1}^{N+1} \left[(F_{N-j+2} \cap \overline{F_{N-j+1}}) \cap \bigcap_{k=0}^{N-j+1} \overline{R_k} \right] \tag{16}$$

where for simplified notation, the event $F_{N+1} \equiv F_t$ is put equal to the failure event at the evaluation time. In Eq. (16), the intersection of consecutive failure events is

$$F_{i+1} \cap \overline{F_i} = [D(t_{i+1}^{\text{insp}}) > D^{\text{thres}} \text{AND} D(t_i^{\text{insp}}) < D^{\text{thres}}] \tag{17}$$

In Eq. (16), it is assumed that any event where the subscript is ≤ 0 is the empty set $\emptyset$. So, for example, the event $\overline{R_k}$ obtained for $k = 0$ is the empty set $\overline{R_0} \equiv \emptyset$.

The consideration of mutually exclusive events, as shown in Eq. (16), leads to the general expression of the total failure probability

$$P[F] = \sum_{j=1}^{N+1} P\left[F_{N-j+2} \cap (\overline{F_{N-j+1}})\right] \prod_{k=0}^{N-j+1} P[(\overline{R_k})] \tag{18}$$

where again for simplicity, the summation goes from 1 to $N + 1$ to include the failure event at the time of observation $F_t \equiv F_{N+1}$.

Eq. (18) could be analytically solved only if both the damage-propagation law of Eq. (3) and the detection probability function of Eq. (3) had a closed-form solution. However, in general, this is not available because the damage-propagation equation is often implicitly solved (for example, using a step forward integration approach), thus the probability of Eq. (18) has to be calculated numerically.

Efficient Forced Monte Carlo Strategy

The computation of reliability is usually associated with quite a significant computational effort. Among the numerical methods proposed in literature, MC simulation methods (Liu 2001; Metropolis and Ulam 1949) are generally applicable but require a compromise between efficiency and accuracy. Many variants of the MC method can be found in literature (Zio and Marseguerra 2002; Schuëller and Pradlwarter 2007), such as line sampling (de Angelis et al. 2015), importance sampling (Au and Beck 1999), and subset simulation (Au and Patelli 2015), which make the MC method more efficient and accurate. Advanced simulation is an essential component of the proposed development to ensure efficiency.

The present numerical strategy is derived from the concept of forced MC simulation described in Zio and Marseguerra (2002). The strategy is based on the computation of weights, w, which account for the probability of detection and can be computed at any inspection time by reusing the results from the same reliability analysis.

Direct Monte Carlo Approach

One way to solve the problem formulated in Eq. (18) is by performing a direct MC simulation.

One Inspection ($N = 1$)

A binary variable, $\delta_{\boldsymbol{\theta}}(t)$, to characterize the outcomes of inspections is introduced. The variable has the following mass function:

$$\delta_{\boldsymbol{\theta}}(t) = \begin{cases} 1(\text{success}) & \lambda_D(t) \\ 0(\text{failure}) & 1-\lambda_D(t) \end{cases} \tag{19}$$

where $\lambda_D = \text{POD}[D_{\boldsymbol{\theta}}(t)]$ is the likelihood of detecting the flaw during inspection.

The total time-variant failure probability is computed by means of MC as

$$p_F(t) = \lim_{N_S\to\infty}\frac{1}{N_S}\sum_{s=1}^{N_S}\int_0^{+\infty}\{\delta_{\boldsymbol{\theta}}(t_1)^{\{s\}}\mathcal{I}(\boldsymbol{\theta},t_1) + [1-\delta_{\boldsymbol{\theta}}(t_1)^{\{s\}}]\mathcal{I}(\boldsymbol{\theta},t)\}h(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{20}$$

where $t_1 = t_1^{\text{insp}}$, and $\delta_{\boldsymbol{\theta}}(t_1)^{\{s\}} \in \{0,1\}$ simulates the outcome of the first inspection for the $\{s\}$th sample. The indicator function in Eq. (20) is

$$\mathcal{I}(\boldsymbol{\theta},t) = \begin{cases} 1, & \text{if } D_q(t) \geq D^{\text{thres}} \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

An extract of the pseudocode that computes the integrand of Eq. (20) is shown in Fig. 1.

The total time-variant failure probability estimator is computed by averaging over a large number of samples, N_S, as

$$\hat{p}_F(t) = \frac{1}{N_S}\sum_{s=1}^{N_S}\{\delta_{\boldsymbol{\theta}}(t_1)^{\{s\}}\mathcal{I}(\boldsymbol{\theta}^{\{s\}},t_1) + [1-\delta_{\boldsymbol{\theta}}(t_1)^{\{s\}}]\mathcal{I}(\boldsymbol{\theta}^{\{s\}},t)\} \tag{22}$$

```
begin
D(t^insp) = D(θ, t^insp)      % compute crack length at inspection time
if D(t^insp) ≥ D^thres then
  I = 1                       % failure occurs before inspection
else
  λ_D = POD(D(t^insp))        % compute the likelihood of detection
  δ ~ {0, 1}_λD               % simulate the outcome of inspection
  if δ = 0 then
    D(t) = D(θ, t)            % compute crack length at the evaluation time
    if D(t) ≥ D^thres then
      I = 1                   % component has failed
    else
      I = 0                   % component has not failed
    end if
  end if
end if
end
```

Fig. 1. Pseudocode for the failure probability estimator of Eq. (22), case with $N = 1$

Multiple Inspections ($N > 1$)

The method can be extended to multiple inspections as a derivation of Eq. (18). Let $\boldsymbol{\delta}^{\{s\}} = \delta_{\boldsymbol{\theta}}(t_1)^{\{s\}}, \ldots, \delta_{\boldsymbol{\theta}}(t_N)^{\{s\}}$ be the vector of inspection outcomes for the $\{s\}$th sample. The total time-variant failure probability can be calculated as

$$p_F(t) = \lim_{N_S\to\infty}\frac{1}{N_S}\sum_{s=1}^{N_S}\int_0^{+\infty}\left\{\sum_{i=1}^{N}[1-\left(1-\delta_i^{\{s\}}\right)\mathcal{I}(\boldsymbol{\theta},t_i)\mathcal{I}(\boldsymbol{\theta},t_{i+1})] \times\prod_{k=0}^{i-1}(1-\delta_k^{\{s\}}) + \mathcal{I}(\boldsymbol{\theta},t)\prod_{k=1}^{N}\left(1-\delta_k^{\{s\}}\right)\right\}h(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{23}$$

where the time t_{N+1} coincides with the evaluation time as $t_{N+1} = t$, and for $k = 0$ the variable $\delta_{k=0}^{\{s\}} = 0$. The integrand of Eq. (23) can be easily coded by means of nested "if" statements, as shown in the pseudo-code of Fig. 2.

The estimator is computed, again, by averaging over a large, albeit finite, number of samples, N_S, as

$$\hat{p}_F(t) = \frac{1}{N_S}\sum_{s=1}^{N_S}\left\{\sum_{i=1}^{N}[1-\left(1-\delta_i^{\{s\}}\right)\mathcal{I}(\boldsymbol{\theta}^{\{s\}},t_i)\mathcal{I}(\boldsymbol{\theta}^{\{s\}},t_{i+1})] \times\prod_{k=0}^{i-1}(1-\delta_k^{\{s\}}) + \mathcal{I}(\boldsymbol{\theta}^{\{s\}},t)\prod_{k=1}^{N}\left(1-\delta_k^{\{s\}}\right)\right\} \tag{24}$$

Forced Monte Carlo Simulation Approach

Here, a numerical approach is proposed to calculate the reliability conditional to inspections without simulating the inspection outcomes. This constitutes a great advantage because, unlike in the direct case, it is no longer necessary to run a full reliability analysis for every inspection time.

One Inspection ($N = 1$)

The strategy can be derived from Eq. (20), noting that the random variable $\delta(t_1)^{\{s\}}$ can be averaged before the integral is calculated, as

```
begin
D(t^insp) = D(θ, t^insp)   % compute crack length at inspection times
δ ~ {0, 1}_λD              % simulate the outcome of inspections
I = 0                      %initialize indicator function
if D(t_1^insp) ≥ D^thres then
  I = 1                    % failure occurs before first inspection
else
  if δ_1 = 0 then
    if D(t_2^insp) ≥ D^thres then
      I = 1                % component has failed
    else
      if δ_2 = 0 then
        ...
        if δ_N = 0 then
          D(t) = D(θ, t)   % compute crack length at the evaluation time
          if D(t) ≥ D^thres then
            I = 1          % component has failed
          else
            I = 0          % component has not failed
          end if
        end if
        ...
      end if
    end if
  end if
end if
end
```

Fig. 2. Pseudocode for the failure probability estimator of Eq. (22) for the case of multiple inspections ($N > 1$)

$$p_F(t) = \int_0^{+\infty} \lim_{N_S\to\infty} \frac{1}{N_S} \sum_{s=1}^{N_S} \{\delta(t_1)^{\{s\}} \mathcal{I}(\boldsymbol{\theta}, t_1) + [1-\delta(t_1)^{\{s\}}]\mathcal{I}(\boldsymbol{\theta}, t)\} h(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (25)$$

which is equivalent to averaging over the inspection outcomes before the reliability analysis is actually performed. Subsequently, the limit is equal to the expected value of the detection probability as

$$w(t) = \lim_{N_S\to\infty} \frac{1}{N_S} \sum_{s=1}^{N_S} \delta(t)^{\{s\}} = \mathrm{POD}[D(\boldsymbol{\theta}, t)] \quad (26)$$

By introducing the weight w from Eq. (26), the total time-variant failure probability becomes

$$p_F(t) = \int_0^{+\infty} \{w(t_1)\mathcal{I}(\boldsymbol{\theta}, t_1) + [1-w(t_1)]\mathcal{I}(\boldsymbol{\theta}, t)\} h(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (27)$$

and the MC estimator is

$$\hat{p}_F(t) = \frac{1}{N_S} \sum_{s=1}^{N_S} \{\hat{w}(t_1)\mathcal{I}(\boldsymbol{\theta}^{\{s\}}, t_1) + [1-\hat{w}(t_1)]\mathcal{I}(\boldsymbol{\theta}^{\{s\}}, t)\} \quad (28)$$

The weight, w, is the relative frequency (relative to the number of runs N_S) by which the logical statement of Fig. 1 if $\delta = 1$ returns true response. This relative frequency, for $N_S \to \infty$ converges to the likelihood of detection $\lambda_D(t_1) = \mathrm{POD}[D(\boldsymbol{\theta}, t_1)]$.

Multiple Inspections ($N > 1$)

The approach is generalized to multiple inspections by computing the detection weights from Eq. (26) by referring to the ith inspection as

$$w_i = \lim_{N_S\to\infty} \frac{1}{N_S} \sum_{s=1}^{N_S} \delta_i^{\{s\}} = \mathrm{POD}[D(\boldsymbol{\theta}, t_i)] \quad (29)$$

Again, by averaging over the inspection outcomes before the integral of Eq. (23) and substituting the weights of Eqs. (29) and (23) it becomes

$$p_F(t) = \int_0^{+\infty} \left\{ \sum_{i=1}^{N} \mathcal{I}(\boldsymbol{\theta}, t_i)[1-(1-w_i)\mathcal{I}(\boldsymbol{\theta}, t_{i+1})] \times \prod_{k=0}^{i-1}(1-w_k) + \mathcal{I}(\boldsymbol{\theta}, t)\prod_{k=1}^{N}(1-w_k) \right\} h(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (30)$$

where for simplified notation $t_{N+1} \equiv t$. The probability estimator is obtained as

$$\hat{p}_F(t) = \frac{1}{N_S} \sum_{s=1}^{N_S} \left\{ \sum_{i=1}^{N} \mathcal{I}(\boldsymbol{\theta}^{\{s\}}, t_i)[1-(1-\hat{w}_i)\mathcal{I}(\boldsymbol{\theta}^{\{s\}}, t_{i+1})] \times \prod_{k=0}^{i-1}(1-\hat{w}_k) + \mathcal{I}(\boldsymbol{\theta}^{\{s\}}, t)\prod_{k=1}^{N}(1-\hat{w}_k) \right\} \quad (31)$$

An alternative and more efficient way to compute Eq. (31), which can be vectorized and hence computed without a *for* loop over the number of samples N_S, is

$$\hat{p}_F(t) = \frac{1}{N_S} \sum_{s=1}^{N_S} \left\{ \mathcal{I}(\boldsymbol{\theta}^{\{s\}}, t) \prod_{j=1}^{N} 1 - \hat{w}_j[1-\mathcal{I}(\boldsymbol{\theta}^{\{s\}}, t_j)] \right\} \quad (32)$$

Total and Partial Probability of Repair

By means of the direct approach, the probability of repair can be calculated for the ith inspection as

$$p_{R_i} = \lim_{N_S\to\infty} \frac{1}{N_S} \sum_{s=1}^{N_S} \prod_{k=1}^{i} \delta_k^{\{s\}} \quad (33)$$

Again, by inverting the order of summation and product sequence, the probability of repair for the ith inspection can be estimated as

$$p_{R_i} = \lim_{N_S\to\infty} \prod_{k=1}^{i} \frac{1}{N_S} \sum_{s=1}^{N_S} \delta_k^{\{s\}} = \prod_{k=1}^{i} E[w_k] \quad (34)$$

The estimator for the total probability of repair, i.e., after all the inspections have been performed, is obtained with a finite sample set as

$$\hat{p}_R = \prod_{k=1}^{N} \hat{w}_k \quad (35)$$

Accuracy of the Proposed Forced Monte Carlo Method

Forced Monte Carlo strategies are ultimately deployed to increase the efficiency of the analysis given a certain accuracy. The MC estimators of Eqs. (24) and (31) are unbiased. The variance of the estimator of Eq. (32) for a fixed number of inspections, N, is directly related to the variance of the estimated failure probability with zero inspections at time t. For example, considering one inspection, $N = 1$, the variance of the probability estimator can be calculated as

$$\mathrm{Var}[\hat{p}_F] = \mathrm{Var}[\mathcal{I}_t] + E[w_1^2](E[\mathcal{I}_t] - E[\mathcal{I}_1]) + E[w_1]^2(E[\mathcal{I}_1]^2 - E[\mathcal{I}_t]^2) \quad (36)$$

where $\mathcal{I}_t = \mathcal{I}(\boldsymbol{\theta}^{\{s\}}, t)$ is the indicator function of Eq. (32). Therefore, the variance of the estimator of the total failure probability is directly related to $\mathcal{I}_t = p_{F0}(t)$, which is the variance of the estimator of the failure probability with zero inspections at time t. This comes with the advantage that $p_{F0}(t)$ is usually very large and therefore very few samples are necessary to achieve a high accuracy.

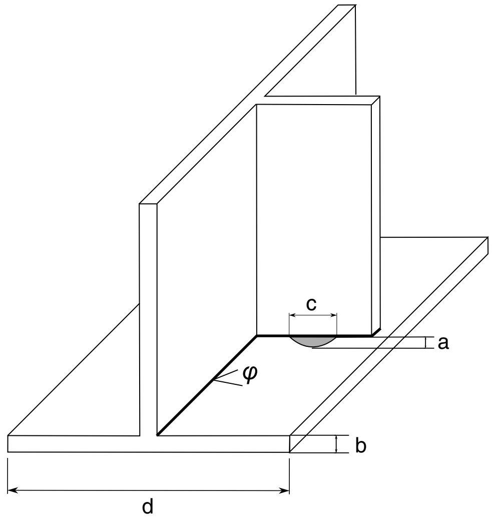
Numerical Example

In this section a welded connection of a bridge girder is analyzed. Due to cyclic loading, metallic components tend to develop fatigue cracks. As these cracks propagate, the structural system accumulates damage that may lead to loss of serviceability or even to collapse, which are followed by considerable monetary losses.

The welded connection between a web stiffener and the girder's flange is studied (Fig. 3). It is assumed that due to stress concentration, a crack appears at the weld's toe. The crack propagation phenomenon is modelled using the Paris-Erdogan law by means of

$$\frac{da}{dT} = C(\Delta K)^m \quad (37)$$

This example has been previously studied in Lukic and Cremona (2001) and Patelli et al. (2013).



F3:1 **Fig. 3.** Structural detail and crack

Uncertainties

The aspect ratio between crack width and depth is assumed to be $a/c = 0.6$. The initial crack length a_0 is given as a lognormal random variable of mean $\mu = 2.0$ mm and standard deviation $\sigma = 0.4$ mm. In this example, time is expressed as the number of fatigue cycles, which can be translated into physical time as the load time pattern is identified. The mission time is $T_M = 2.5 \times 10^6$ cycles. For the critical crack length, a_c, a Gumbel distribution is considered, while for the parameters of the Paris law, m and $\ln C$, normal distributions are considered. A detailed justification of the formulated assumptions on the preceding probability distributions can be found in Lukic and Cremona (2001). Numerical values, distribution functions, and units are provided for all parameters in Table 1.

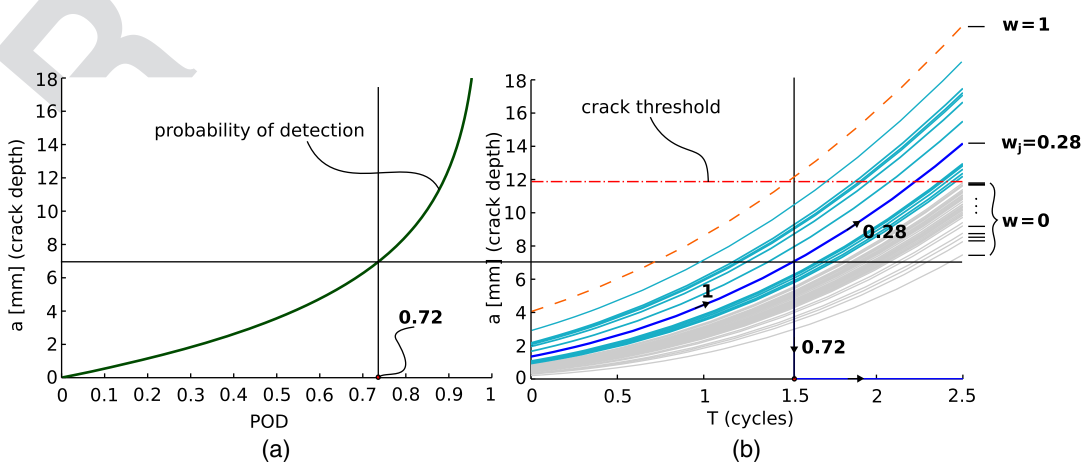
Estimation of Total Failure Probability with One Inspection

When just one single inspection is performed, the problem can be visualized, as shown in Fig. 4. The time-variant failure probability is computed with one inspection performed at time $t^{\text{insp}} = 1.5 \times 10^6$. For the purpose of illustration, Fig. 4 is produced with the uncertainty assigned to the initial crack length, a_0, only. This is done to better visualize the curves of Fig. 4 and actually see the curves corresponding to each single simulation beyond the specified critical threshold. The rest of the numerical example is solved with uncertainties defined as in Table 1. The reliability analysis generates a bundle of curves (Fig. 4), where each curve is obtained solving the crack-growth equation for each sample.

Fig. 4 illustrates the procedure to calculate the weights using the forced Monte Carlo simulation approach. Fig. 4(a) shows the probability of detection (x-axis) as a function of the level of damage. For every sample of the reliability analysis a curve of the level of damage, a, is plotted as a function of time. At the end of the reliability analysis the graph in Fig. 4(b) is obtained. In order to calculate the weights, a line is drawn through both graphs corresponding to the POD at time $t = 1.5 \times 10^6$. The picked curve in Fig. 4(b) is tracked with probability 1 until inspection occurs. Following inspection there are two possible states with associated probabilities: either the detection is successful (with probability 0.72) and the damage is removed $a(t \geq t^{\text{insp}}) = 0$, or the detection fails and the damage keeps growing until the evaluation time $t = 2.5 \times 10^6$. The probability of nondetection (0.28) is the weight to be assigned to the sample under consideration. The procedure is

Table 1. Numerical Values, Units, and Distribution Functions for the Input and Model Parameters of the Numerical Example

	θ_i	$E[\theta_i]$	$\mathrm{Var}[\theta_i]$	Distribution	Units	Description
T1:2	a_0	2.0	0.4^2	Lognormal	mm	Initial crack length
T1:3	a_c	15	0.025^2	Gumbel	mm	Critical threshold
T1:4	m	2.4	0.024^2	Normal	—	Exponent of Eq. (37)
T1:5	C	2×10^{-10}	$(10^{-12})^2$	Lognormal	mm/cycles	Coefficient of Eq. (37)
T1:6	$\Delta\sigma$	30	0.1^2	Gamma	MPa	Fatigue stress range
T1:7	a/c	0.6	0	Deterministic	—	Crack shape
T1:8	p_0	0.02	0	Deterministic	—	POD's factor
T1:9	f_s	0.1	0	Deterministic	1/mm	POD's factor



F4:1 **Fig. 4.** Diagram for the calculation of weights and failure probability conditional to the outcome of one inspection

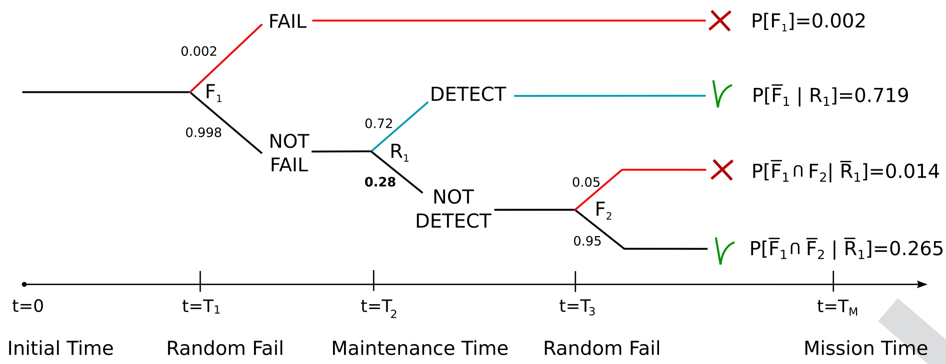


Fig. 5. Tree chart for the calculation of the failure probability conditional to the outcome of one inspection

then repeated for every curve that fails at the evaluation time (thick lines), except for those curves that have already failed at the inspection time, which samples are given weight $w = 1$. The chronological flow pursued to calculate the weights and to obtain the failure probability is represented in Fig. 5

The total failure event at the time of observation, as shown in Fig. 5, is obtained as

$$F = F_1 \cup (F_t \cap \overline{F_1} | \bar{R}) \tag{38}$$

and, consequently, the total failure probability can be calculated as

$$P[F] = P[F_1] + P[F_t \cap \overline{F_1}]P[\bar{R}] \tag{39}$$

where $P[\bar{R}] = 1 - w$ is the probability of nondetection or nonrepair; and $F_t = [a(t) > a_c]$ is the failure event at the evaluation time. The value of total failure probability obtained from Fig. 5 is $P[F_1] + P[F_t \cap \overline{F_1}]P[\bar{R}] = 0.002 + 0.014 = 0.016$.

Design of Maintenance Strategies with One and Two Inspections

The optimization is formulated as in "Optimization of Maintenance Costs," where the inspection time $\boldsymbol{t}^{\text{insp}}$ and the inspection quality q are the design variables of the problem. The input unit costs of maintenance and failure are reported in Table 2.

Fig. 6 shows how the total cost displays on a graph as a function of the inspection time because the inspection quality is fixed as $q = 1$. Although the cost of failure may appear quite smooth, every point of the curve is obtained by estimating the failure probability and therefore it is associated with an, albeit small, estimation error. The curve of failure cost shows a typical concave shape. The minimum is located at $t^{\text{insp}*} = 2.0 \cdot 10^6$. The minimum total cost is slightly shifted because the cost of inspection decreases with the inspection time. The cost of repair is very small compared to the other costs (Fig. 6).

With two inspections, the total cost is represented on a time grid, where every point corresponds to a pair $(t_i^{\text{insp}}, t_j^{\text{insp}})$. The resulting graph is the surface in Fig. 7. The minimum total cost identifies the pair $\boldsymbol{t}^{\text{insp}*} = (1.87510^6, 2.08510^6)$ when the two inspections should be performed. Fig. 8 shows the costs of repair, which are significantly smaller compared to the total costs.

Solution to the Constrained Cost Optimization with N Inspections

It is recalled that the number of design variables is $N + 1$ because the inspection quality q is assumed to be equal for all inspections.

Table 2. Unit Cost Used in the Numerical Example

Cost	Value	Description
c_I	30,000	Unit cost of inspection
c_R	500	Unit cost of repair
c_F	2×10^5	Unit cost of failure
r	0.05	Discount rate

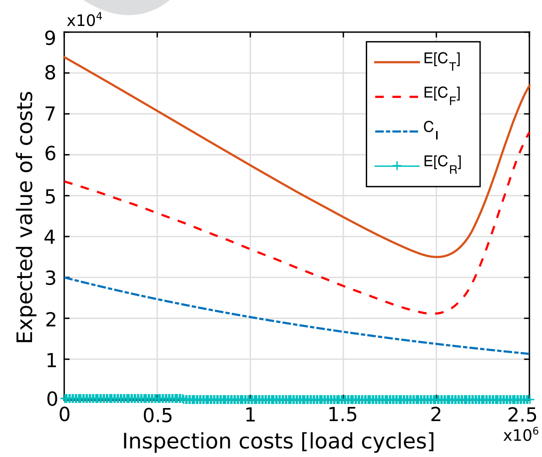


Fig. 6. Curves of cost obtained fixing the quality of inspection to $q = 1$

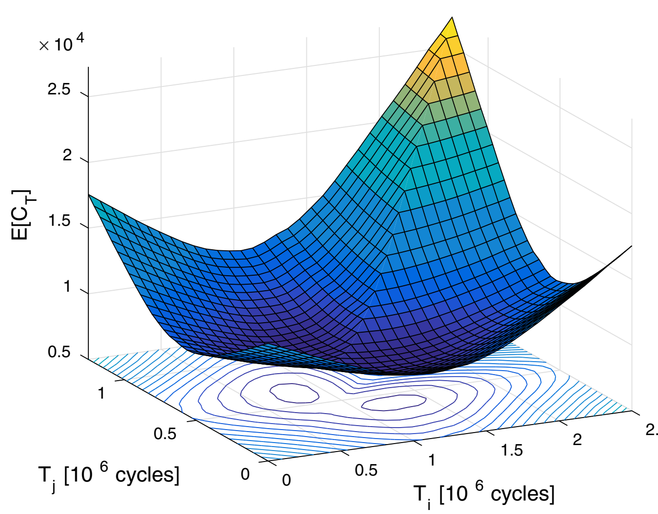


Fig. 7. Total costs surface with two inspections $N = 2$

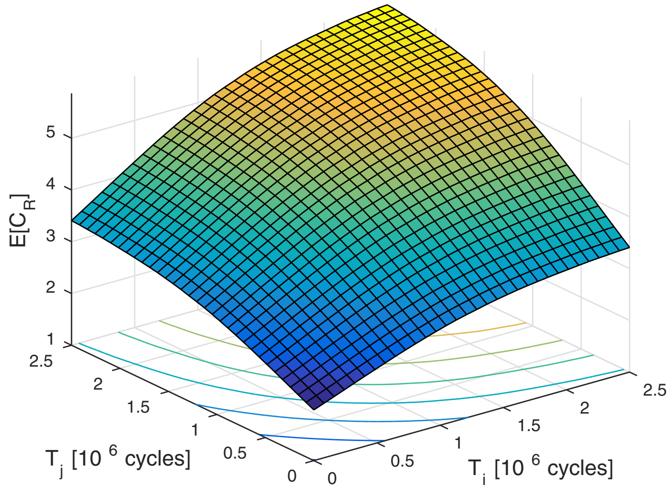


Fig. 8. Repair cost surface with two inspections $N = 2$

The solution to this problem is found using a genetic algorithm (GA) (Goldberg 2006). The choice of GA is justified by the stochastic nature of the objective function and constraint, which do not behave as smooth functions. As a consequence, information about the derivatives on both objective and constraint cannot be exploited, and, moreover, the presence of narrow spikes, albeit on a small scale, makes ineffective the use of any derivative-based optimizers. Any other optimizer that does not make use of derivatives could have been used alternatively. Results from the constrained optimization as the number of inspections increases are obtained fixing the critical failure probability to $p_F^{\text{critic}} = 10^{-3}$. The variance of the estimator changes from case to case, depending on the failure probability target, however, for this problem it can be associated with a coefficient of variation (COV) approximately equal to 10^{-3}. The computational cost of performing a complete analysis is directly related to the total number of samples, N_S. With the proposed approach, the number of samples required by the reliability analysis only, n_S, is decoupled from the optimization procedure. In this way the optimizer does not need to sample from the full model to proceed with the iterations and the total number of samples is $N_S = n_S$. On the contrary, using a plain Monte Carlo approach, the number of samples is given by $N_S = N \cdot N_{\text{opt}} \cdot n_S$, where N_{opt} is the number of objective (total cost) function evaluations and N is the number of inspections. In fact, with such an approach, every objective function evaluation requires as many reliability analyses as the number of inspections, making the plain Monte Carlo approach ever more inefficient as the number of inspections grows. Results of Table 3 are obtained with the forced Monte Carlo approach using $N_S = n_S = 10^5$ in total. With the plain approach, using $n_S = 10^5$, a complete cost-benefit analysis would have required $N_S = N \cdot N_{\text{opt}} \cdot 10^5$, where the number of objective evaluations in this case required by the optimizer is approximately $N_{\text{opt}} \simeq 2.5 \times 10^3$. Therefore, depending on how many inspections are performed (N), the total number of samples required by the plain approach can be as large as $N_S \simeq 10^{10}$. Provided the failure probability with zero inspections at time t is approximately $p_{F0} \simeq 0.15$, the coefficient of variation of the probability estimator can approximated using Eq. (36) neglecting the second and third terms of the left-hand side. Given the number of samples used in the reliability analysis $n_S = 10^5$, the coefficient of variation of the estimator is approximately $\text{COV} \simeq \sqrt{(1 - \hat{p}_{F0})/(n_S \hat{p}_{F0})} = 0.001$, which is a very accurate estimation.

Table 3 shows that there is no sensible decrease in the total cost of maintenance as the number of inspections increases. There is nearly no difference between optima obtained with two, three, four, five, six, and seven inspections, which makes these four options equally suitable. However, the optimum quality decreases from two to seven inspections, which means that at the same total cost, less expensive inspection techniques could be used. Performing more than seven inspections does not bring an appreciable reduction to the total cost, thus there is no advantage in increasing the number of inspections beyond seven unless a smaller critical value of failure probability is fixed. In Table 3, results obtained with one inspection are omitted because the required constraint is not satisfied.

It is also interesting to see how the total costs drastically decrease as the constraint is removed from the optimization compared with the price of much larger failure probabilities (Table 4). By removing the failure probability constraint ($p_F^{\text{critic}} < 10^{-3}$), the total cost of maintenance decreases by a factor of ~10. From Table 4, it can also be seen that when no constraint is imposed, the total cost does not sensibly change as the number of inspections increases.

Table 3. Minimum Total Costs and Associated Arguments as the Number of Inspections Increases for the Case $p_F^{\text{thres}} = 10^{-3}$

N	$\hat{p}_F$	$\min C_M$	q^*	$t_1^{\text{insp}*}$	$t_2^{\text{insp}*}$	$t_3^{\text{insp}*}$	$t_4^{\text{insp}*}$	$t_5^{\text{insp}*}$	$t_N^{\text{insp}*}$
2	9.55×10^{-4}	1.35×10^5	4.47	2.75×10^6	3.39×10^6	—	—	—	—
3	7.21×10^{-4}	1.18×10^5	2.68	1.73×10^6	1.83×10^6	1.95×10^6	—	—	—
4	9.33×10^{-4}	1.11×10^5	1.95	1.62×10^6	1.83×10^6	2.00×10^6	2.22×10^6	—	—
5	9.00×10^{-4}	1.34×10^5	1.94	1.71×10^6	1.88×10^6	2.03×10^6	2.06×10^6	2.13×10^6	—
6	9.32×10^{-4}	1.11×10^5	1.31	1.71×10^6	1.78×10^6	1.84×10^6	1.96×10^6	2.10×10^6	2.36×10^6
7	7.44×10^{-4}	1.11×10^5	1.12	1.48×10^6	1.76×10^6	1.83×10^6	1.95×10^6	1.99×10^6	$\ldots 2.32 \times 10^6$
8	8.19×10^{-4}	1.73×10^5	1.61	1.46×10^6	1.75×10^6	1.99×10^6	2.05×10^6	2.21×10^6	$\ldots 2.49 \times 10^6$
9	9.70×10^{-4}	1.72×10^5	1.44	1.63×10^6	1.85×10^6	1.89×10^6	2.08×10^6	2.25×10^6	$\ldots 2.35 \times 10^6$
10	9.10×10^{-4}	1.36×10^5	0.99	1.60×10^6	1.83×10^6	1.84×10^6	2.00×10^6	2.06×10^6	$\ldots 2.33 \times 10^6$

Table 4. Minimum Total Costs and Associated Arguments as the Number of Inspections Increases for the Unconstrained Case ($p_F^{\text{thres}} = 1$)

N	$\hat{p}_F$	$\min C_M$	q^*	$t_1^{\text{insp}*}$	$t_2^{\text{insp}*}$	$t_3^{\text{insp}*}$	$t_4^{\text{insp}*}$
1	2.15×10^{-1}	3.43×10^4	1.35	2.07×10^6	—	—	—
2	1.32×10^{-1}	3.33×10^4	0.85	1.92×10^6	2.06×10^6	—	—
3	1.21×10^{-1}	3.36×10^4	0.56	1.94×10^6	2.04×10^6	2.07×10^6	—
4	2.70×10^{-1}	3.67×10^4	0.28	1.55×10^6	1.69×10^6	2.02×10^6	2.05×10^6

However, as in the constrained case, a greater number of inspections may allow reducing the quality to the same total maintenance cost.

Within the proposed framework and when N is large, inspections can be performed quite early. This is justified by the fact that the time zero is a relative time and does not necessarily coincide with a newly crafted component. This is in line with the underlying fracture mechanics model adopted to predict the crack length of the components, which assumes a crack initiation process. Results from the numerical example show how the number of inspections, N, does not significantly influence the minimum total cost. However, a change in the inspection quality from $N = 2$ to $N = 4$ can be appreciated, meaning that less expensive inspection techniques can be used at the same total cost. Results from the optimization also show that there is no significant advantage in performing more than seven inspections because the total cost slightly increases and the optimal quality does not change in value. By looking at the results from the unconstrained case, similar conclusions can be drawn because the total cost does not change and the inspection quality decreases with the number of inspections.

Conclusions

The solution to the maintenance problem is a trade-off between costs associated with inspection and repair activities and benefits related to the faultless operation of the infrastructure. The maintenance of a system is a complex engineering task, where the estimation of costs requires the consideration of heterogeneous uncertainties arising from the damage propagation process and from the inspection and repair activities. The maintenance problem is turned into an optimization problem, which is regarded as a stochastic and discrete optimization problem, involving significant computational cost.

A general and efficient methodology for the robust design of maintenance plans has been presented. The methodology exploits the concept of forced MC simulation to maximize the efficiency without reducing its accuracy. The strategy makes use of the computation of weights that emulate the outcomes of the preventive maintenance. The strategy is capable of assessing the reliability of the system at any given time and for any number of inspections, performing only one full reliability analysis, whose results can subsequently be used with nearly no additional computational effort.

The computational efficiency is directly related to the number of samples required to complete the optimization. The number of samples used by the proposed approach equalizes the number of samples required to complete a single reliability analysis, $N_S = n_S$. A plain Monte Carlo approach would instead require many more samples $N_S = N \cdot N_{\text{opt}} \cdot n_S$, where N is the number of inspections and N_{opt} is the number of objective function evaluations.

The proposed methodology has been derived from a direct Monte Carlo strategy and therefore has no limitations in terms of number of uncertain variables. Moreover, it is easy to implement and parallelize, allowing its application to real-scale systems of industrial interest. The numerical strategy has been implemented in OpenCossan (Patelli et al. 2014) making it directly accessible and applicable for industrial research.

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